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The Gaussian approximation for interacting charged scalar fields

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Abstract. The Gaussian effective-potential approach is used to explore the physics of charged $\lambda(\phi^+\phi)^2$ theory in four space-time dimensions. We find and employ an appropriate trial system, parametrized by two effective masses, for obtaining an adequate Gaussian effective potential under conditions of the global U(1) symmetry and the finite temperature. A simple renormalization, accompanied by an explicit dimensional regularization, is employed. We find that the nontrivial approach arises from a bare coupling constant of a negative infinitesimal form, well known in the noncharged case as "precarious". The behavior of this solution is discussed, and the symmetry breaking due to background charge density is discovered.

1 Introduction

The Gaussian approximation is a nonperturbative method, proposed at the beginning of the 1960s [1], but largely used in the study of interacting fields, especially after the proposal of the Gaussian effective potential (GEP) [2]. It has been shown that the GEP reveals the properties of the ground state of quantum systems and is superior to the usual one-loop effective potential. In the last decade, the Gaussian method has been developed largely in the study of interacting fields at zero [3–6] and at finite temperature [7–10].

Our aim is to use the Gaussian approximation for the study of the four-dimensional complex scalar fields in $\lambda(\phi^+\phi)^2$ interaction and at finite temperature. So, unlike previous studies of real $\lambda\phi^4$ interacting fields, we find and make use of a more fertile trial configuration, parametrized by two effective masses.

We use the dimensional regularization for removing the divergences. Such a procedure allows us to find the so-called precarious solution, which shows a symmetry breaking of the Bose–Einstein-condensation type, due to background charge. The organization of the paper is as follows: In the Sect. 2,we attempt to find a convenient GEP for our initial system $\lambda(\phi^+\phi)^2$. First, we explain the importance of an effective Lagrangian under the conditions of charge conservation and finite temperature. After having found that Lagrangian, we are led to define and study a generalized configuration parametrized by two different effective masses. For that system, we find the effective potential and the spectrum. We use the functional formulation in terms of Feynman path integrals, where the fields are periodic in imaginary time. This turns out to be the only way in the case of our special trial model. For arriving at the GEP of the initial theory, we continue to use the functional formulation. The effective parameters are chosen in such a way that the GEP should be as close as possible to the generating function of Green's functions. In Sect. 3, we make use of the dimensional regularization for removing the divergences. Using renormalization, we eliminate the bare parameters m^2 and λ in favor of two finite parameters m_R and λ_R and find the precarious case. The last section is devoted to the analysis of the precarious case, especially for low temperatures, where the differences with the noncharged corresponding case are crucial. This solution shows a symmetry breaking due to charge conservation.

2 Gaussian effective potential (GEP)

2.1 The effective Lagrangian

Consider a self-interacting charged scalar field with the Lagrangian density

$$\mathcal{L} = \partial^\mu\phi\partial_\mu\phi^+ - m^2\phi\phi^+ - \frac{\lambda_0}{4!}(\phi\phi^+)^2, \tag{1}$$

which corresponds to a Hamiltonian with density

$$\mathcal{H} = \pi\pi^+ + \nabla\phi\nabla\phi^+ + m^2\phi\phi^+ + \frac{\lambda_0}{4!}(\phi\phi^+)^2; \tag{2}$$

ϕ, ϕ^+ represent the fields and π, π^+ their associated momenta. There is a global U(1) symmetry which leads to the conservation of the charge density q, so the partition function at finite temperature and in the presence of the external sources J, J^* is given by the functional integral

$$\begin{aligned}Z &= \mathrm{Tr}\left\{\exp(-\beta\mathcal{H} + \mu q)\right\} \\ &= N\int \mathrm{D}\pi\mathrm{D}\pi^+ \int_{\text{period}} \mathrm{D}\phi\mathrm{D}\phi^+ \\ &\quad\times \exp\left\{\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \Big[\mathrm{i}\pi^+\frac{\partial\phi}{\partial\tau} + \mathrm{i}\pi\frac{\partial\phi^+}{\partial\tau} \right. \\ &\quad\left. -\mathcal{H}(\pi^+,\pi,\phi^+,\phi) + \mu q(\pi^+,\pi,\phi^+,\phi) + J^*\phi + J\phi^+\Big]\right\}.\end{aligned} \tag{3}$$

Here μ is the chemical potential, β the temperature inverse, and τ the imaginary time, obtained by the Wick rotation $t \to -\mathrm{i}\tau$. The field integration is restricted to periodic orbits; the momentum integration is over all possible π. The procedure which allows one to write the partition function as a functional integral over field configurations has been discussed extensively by Bernard [11] and especially for charged scalar fields by Kapusta [12].

A quite useful form of the partition function is realized if one replaces ϕ, ϕ^+ by real fields ϕ_1 and ϕ_2,

$$\phi_1 = \frac{\phi + \phi^+}{\sqrt{2}}, \qquad \phi_2 = \frac{\phi - \phi^+}{\sqrt{2}\mathrm{i}}, \tag{4}$$

and integrates over momenta. The result is then[1]

$$Z = N(\beta) \int_{\text{period}} \mathrm{D}\phi_1 \mathrm{D}\phi_2 \times \exp\left\{ \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\mathcal{L}_{\text{eff}}^{\mathrm{E}} + J_a\phi_a \right] \right\}, \tag{5}$$

with

$$\mathcal{L}_{\text{eff}}^{\mathrm{E}} = -\frac{1}{2}\left\{ \frac{\partial\phi_a}{\partial\tau}\frac{\partial\phi_a}{\partial\tau} + \nabla\phi_a\nabla\phi_a + (m^2-\mu^2)\phi_a\phi_a \right\} -\frac{\lambda}{4!}(\phi_a\phi_a)^2 - \mathrm{i}\mu\left(\phi_1\frac{\partial\phi_2}{\partial\tau} - \phi_2\frac{\partial\phi_1}{\partial\tau} \right). \tag{6}$$

Instead of J and J^* we have the sources J_a: J_1, J_2. $a = 1,2$ indicates also the sum, when repeated, for an elegant representation; we denote by $\lambda \equiv \lambda_0/4$.

In the last expression of the partition function, $N(\beta)$ is an infinite coefficient of normalization; it results from the integration over momenta and has been calculated by Bernard [11]. $\mathcal{L}_{\text{eff}}^{\mathrm{E}}$ is the effective Lagrangian in Euclidian space-time. Written in Minkowskian space-time by the change $\partial/\partial\tau \to -\mathrm{i}\partial/\partial t$:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\left\{ \partial^\nu\phi_a\partial_\nu\phi_a + (\mu^2 - m^2)\phi_a\phi_a \right\} -\frac{\lambda}{4!}(\phi_a\phi_a)^2 - \mu\left(\phi_1\partial_0\phi_2 - \phi_2\partial_0\phi_1\right). \tag{7}$$

This differs from the initial Lagrangian, which in terms of ϕ_1, ϕ_2 would be

$$\mathcal{L} = \frac{1}{2}\left\{ \partial^\nu\phi_a\partial_\nu\phi_a - m^2\phi_a\phi_a \right\} - \frac{\lambda}{4!}(\phi_a\phi_a)^2. \tag{8}$$

Notice that the difference between the ordinary and the effective Lagrangian, apart from the term μq ($q = \phi_1\partial_0\phi_2 - \phi_2\partial_0\phi_1$, see [12]), is the amount $\mu^2\phi_a\phi_a$. Obviously, the effective Lagrangian replaces the original Lagrangian in calculation of the partition function, under conditions of charge conservation.

Let us write, in terms of the effective Lagrangian, the generating functional $W[J_1, J_2]$ of Green's functions

$$e^{W[J_1,J_2]} = Z[J_1,J_2] = N(\beta)\int_{\text{period}} \mathrm{D}\phi_1\mathrm{D}\phi_2 \tag{9}$$
$$\times \exp\left\{ \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\mathcal{L}_{\text{eff}}^{\mathrm{E}}(\phi_1,\phi_2) + J_a\phi_a \right] \right\}.$$

By the Legendre transformation $J \to \sigma$, the effective potential $V(\sigma,\mu)$ would then be

$$\exp[-\beta V(\sigma,\mu)] = \exp\left[W[J_1,J_2] - \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, J_a\sigma_a \right], \tag{10}$$

where σ_1, σ_2 are the c-number fields implicated by J_1, J_2: $\sigma_a = \mathrm{d}W/\mathrm{d}J_a$. One can find an extensive discussion about the last two mathematical relations in [13].

Carrying out a shift $\phi_a \to \sigma_a + \phi_a$, the effective potential can be rewritten in a more useful form,

$$\exp[-\beta V(\sigma,\mu)] = N(\beta)\int \mathrm{D}\phi_1\mathrm{D}\phi_2 \times \exp\left\{ \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\mathcal{L}_{\text{eff}}^{\mathrm{E}}(\sigma+\phi) + \phi_a\frac{\mathrm{d}(\beta V)}{\mathrm{d}\sigma_a} \right] \right\}, \tag{11}$$

where the relation $J_a = \mathrm{d}(\beta V)/\mathrm{d}\sigma_a$ has been used.

2.2 A generalized quadratic model

Inspired by (7), let us study the generalized effective Lagrangian density

$$\mathcal{L}_{\text{free}} = \frac{1}{2}\left\{ \partial^\nu\phi_a\partial_\nu\phi_a + (\mu^2 - m_1^2)\phi_1^2 + (\mu^2 - m_2^2)\phi_2^2 \right\} -\mu\left(\phi_1\partial_0\phi_2 - \phi_2\partial_0\phi_1\right) - \alpha\phi_1\phi_2 \tag{12}$$

of the quasi-particle system with two different masses and a coupling coefficient α. It will serve us below as a trial model in calculating the GEP.

After the shift $\phi_a \to \phi_a + \phi_{a0}$, the quasi-particle's partition function (5) would be:

$$\begin{aligned} Z_{\text{free}}[J_1,J_2] &= N(\beta)\int \mathrm{D}\phi_1\mathrm{D}\phi_2 \\ &\times \exp\left\{ \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, \mathcal{L}_{\text{free}}^{\mathrm{E}}(\phi_1,\phi_2) \right\} \\ &\times \exp\left\{ \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[(\mu^2 - m_a^2)\phi_a\phi_{a0} \right.\right. \\ &\left.\left. -\alpha(\phi_{10}\phi_2 + \phi_{20}\phi_1) + J_a(\phi_a + \phi_{a0}) \right]\right\} \\ &\times \exp\left\{ \beta\left(\int d^3x\right)\left[\frac{1}{2}(\mu^2 - m_1^2)\phi_{10}\phi_{10} \right.\right. \\ &\left.\left. +\frac{1}{2}(\mu^2 - m_2^2)\phi_{20}\phi_{20} - \alpha\phi_{10}\phi_{20} \right]\right\}, \end{aligned} \tag{13}$$

where $\mathcal{L}_{\text{free}}^{\mathrm{E}}$ is the expression (12) written in Euclidian space-time.

The first exponential function in the above functional integration represents the quantum and thermal quasi-particle fluctuations around ϕ_{a0}, the second depends linearly on

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[1] See [12] for a similar result.

ϕ_1, ϕ_2, and the third is a constant. An appropriate choice of ϕ_{a0} cancels the second exponential, and the partition function now reads

$$Z_{\rm free}(J_1,J_2) = e^{W_0}\exp\left\{\frac{\beta}{2}\frac{(\mu^2-m_2^2)J_1^2+(\mu^2-m_1^2)J_2^2-2\alpha J_1J_2}{(\mu^2-m_1^2)(\mu^2-m_2^2)-\alpha^2}\right\}. \quad (14)$$

e^{W_0} is the result of the functional integration of the first exponential in (13). Carrying out the Legendre transformation $J_a \to \sigma_a$, we are immediately led to the effective potential

$$V_{\rm free}(\sigma) = V_0 + \frac{1}{2}\left\{(\mu^2-m_1^2)\sigma_1^2 +(\mu^2-m_2^2)\sigma_2^2-2\alpha\sigma_1\sigma_2\right\} \quad (15)$$

where $V_0 = -W_0/\beta$, or

$$V_0 \equiv -\frac{1}{\beta}\ln Z_0 = -\frac{1}{\beta}\ln N(\beta)\int {\rm D}\phi_1{\rm D}\phi_2 \exp\left\{\int_0^\beta {\rm d}\tau\int {\rm d}^3x\,\mathcal{L}^{\rm E}_{\rm free}(\phi_1,\phi_2)\right\}. \quad (16)$$

The fields ϕ_1 and ϕ_2 may be expanded as Fourier series in the range $[0,\beta]$, where they are periodic:

$$\phi_a(\vec{x},\tau) = \frac{1}{\beta}\sum_n\int\frac{{\rm d}^3\vec{k}}{(2\pi)^3}{\rm e}^{{\rm i}(\vec{k}\cdot\vec{x}+\omega_n\tau)}\phi_a(\vec{k},n), \quad (17)$$

with $\omega_n = 2\pi n/\beta$. Thus, one can find

$$\int_0^\beta {\rm d}\tau\int {\rm d}^3x\,\mathcal{L}^{\rm E}_{\rm free} = -\frac{1}{2\beta}\sum_n\int\frac{{\rm d}^3\vec{k}}{(2\pi)^3}\phi_a(-\vec{k},-n)\mathbf{A}_{ab}\phi_b(\vec{k},n) \quad (18)$$

where

$$\mathbf{A} = \begin{bmatrix}\omega_n^2+\vec{k}^2+m_1^2-\mu^2 & -2\mu\omega_n+\alpha\\ 2\mu\omega_n+\alpha & \omega_n^2+\vec{k}^2+m_2^2-\mu^2\end{bmatrix}.$$

Calculating the last functional integral (16) with the quadratic form (18) one obtains

$$\exp[-\beta V_0] = \exp\left[\ln N(\beta)-\frac{1}{2}\sum_n\int\frac{{\rm d}^3\vec{k}}{(2\pi)^3}\ln({\rm Det}\ \mathbf{A})\right], \quad (19)$$

where

$$\text{Det }\mathbf{A} = \left(\omega_n^2+E_+^2\right)\left(\omega_n^2+E_-^2\right) \quad (20)$$

with

$$E_\pm^2 = \vec{k}^2+\mu^2+\frac{1}{2}(m_1^2+m_2^2) \pm\frac{1}{2}\left[(m_1^2-m_2^2)^2+8\mu^2(2\vec{k}^2+m_1^2+m_2^2)+4\alpha^2\right]^{\frac{1}{2}}. \quad (21)$$

We obtain the spectrum $E_\pm^2$ of our special quasi-particles, found by solving the equation Det $\mathbf{A}=0$ with $\mathbf{A}$ written in Minkowskian space-time (so $\omega_n \to {\rm i}E$).

Using a similar algebra as that in [14,15], we have for V_0:

$$\begin{aligned}V_0 &= \int\frac{{\rm d}^3\vec{k}}{2(2\pi)^3}\left(E_++E_--\sqrt{k^2+m_1^2}-\sqrt{k^2+m_2^2}\right)\\ &\quad+\int\frac{{\rm d}^3\vec{k}}{2(2\pi)^3}\left(\sqrt{k^2+m_1^2}+\sqrt{k^2+m_2^2}\right)\\ &\quad+\int\frac{{\rm d}^3\vec{k}}{\beta(2\pi)^3}\left[\ln(1-{\rm e}^{-\beta E_+})+\ln(1-{\rm e}^{-\beta E_-})\right]\\ &= V_{0\mu}+V_{0v}+V_{0T}.\end{aligned} \quad (22)$$

V_0 is the thermodynamic function, which averages vacuum and thermal fluctuations of the mixed quasi-particle configuration (12) in equilibrium at temperature $T = 1/\beta$. It contains infinities, due to the vacuum contribution V_{0v}, which diverges as k^4. The contribution of the thermal excitations is obviously included at the third finite integral V_{0T}. The first finite integral, $V_{0\mu}$, contains exclusively the charge contribution; it vanishes for $q = 0$.

Notice that for $m_1 = m_2 = m$ and $\alpha = 0$, the simple well-known form

$$V = V_0 + \frac{\mu^2-m^2}{2}(\sigma_1^2+\sigma_2^2) \quad (23)$$

is discovered for (15), with

$$V_0 = \int\frac{{\rm d}^3\vec{k}}{(2\pi)^3}\left[\sqrt{\vec{k}^2+m^2}+\frac{1}{\beta}\ln\left(1-{\rm e}^{-\beta E_+}\right) +\frac{1}{\beta}\ln\left(1-{\rm e}^{-\beta E_-}\right)\right] \quad (24)$$

and the spectrum:

$$E_\pm = \sqrt{\vec{k}^2+m^2}\pm\mu. \quad (25)$$

The above effective potential V obviously has its minima at $\sigma^2 = \sigma_1^2+\sigma_2^2 = 0$, $\mu \neq m$, or at $\sigma^2 \neq 0$, $\mu = m$. The latter possibility represents a symmetry breaking due to Bose– Einstein condensation, a well-known phenomenon studied by several authors. Here we mention [12,15,16] where one can find an analysis of Bose–Einstein condensation for ideal and interacting Bose gas. When the temperature goes down below a critical temperature $T_{\rm crit}$, a macroscopic number of particles condenses on the fundamental state $\sigma^2 \neq 0$. As $\mu = m$, this symmetry breaking is associated with a zero spectrum:

$$E_-(\vec{k}=0) = 0. \quad (26)$$

2.3 Gaussian approximation

Now we search for the GEP of the initial theory (1). Let us choose as trial model the quite general quasi-particle system with Lagrangian density (12), parametrized by m_1, m_2, α. These effective parameters are free to be chosen conveniently. Let us first rewrite the expression (11) in another form:

$$\begin{aligned}&\exp[-\beta V(\sigma,\mu)]\\&= N(\beta)\int \mathrm{D}\phi_1\mathrm{D}\phi_2 \exp\left\{\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x \left[\mathcal{L}^{\mathrm{E}}_{\mathrm{eff}}(\sigma_a+\phi_a)\right.\right.\\&\left.\left.-\mathcal{L}^{\mathrm{E}}_{\mathrm{free}}(\phi_a)+\phi_a\frac{\mathrm{d}(\beta V)}{\mathrm{d}\sigma_a}\right]\right\}\times\exp\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3x\, \mathcal{L}^{\mathrm{E}}_{\mathrm{free}}(\phi_a)\\&=< e^{S-S_{\mathrm{free}}} > Z_0. \qquad (27)\end{aligned}$$

Here $< ... >$ shows the thermodynamic average:

$$< ... >\equiv \qquad (28)$$
$$\frac{N(\beta)\int \mathrm{D}\phi_1\mathrm{D}\phi_2 ... \exp\left\{\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3x\,\mathcal{L}^{\mathrm{E}}_{\mathrm{free}}(\phi_1,\phi_2)\right\}}{Z_0}.$$

The convexity of the exponential function implies $< e^{S-S_{\mathrm{free}}} > \ \geq e^{<S-S_{\mathrm{free}}>}$ for real $(S - S_{\mathrm{free}})$, so

$$\begin{aligned}&e^{-\beta V(\sigma,\mu)} \geq e^{<S-S_{\mathrm{free}}>}Z_0 \qquad \text{or}\\&-\beta V(\sigma,\mu)\geq \ln Z_0 + < S-S_{\mathrm{free}} > . \qquad (29)\end{aligned}$$

In another form, making use of $< S - S_{\mathrm{free}} >= \beta < L - L_{\mathrm{free}} >$, we define and write

$$\begin{aligned}V(\sigma,\mu)&\leq G(m_1,m_2,\alpha)\\&= V_0 + < \mathcal{L}^{\mathrm{E}}_{\mathrm{free}}(\phi)-\mathcal{L}^{\mathrm{E}}_{\mathrm{eff}}(\sigma+\phi) > \qquad (30)\end{aligned}$$

where $G(m_1,m_2,\alpha)$ is a set of functions, defined by (30), parameterized by m_1, m_2, and α. Directly by (12) and (6) we have

$$\begin{aligned}&\mathcal{L}^{\mathrm{E}}_{\mathrm{free}}(\phi)-\mathcal{L}^{\mathrm{E}}_{\mathrm{eff}}(\sigma+\phi)\\&=\frac{m^2-\mu^2}{2}(\sigma_a+\phi_a)^2+\frac{\lambda}{4!}(\sigma_a+\phi_a)^4\\&\quad-\frac{m_1^2-\mu^2}{2}\phi_1^2-\frac{m_2^2-\mu^2}{2}\phi_2^2-\alpha\phi_1\phi_2. \qquad (31)\end{aligned}$$

This last expression contains terms in ϕ, ϕ^2, ϕ^3, and ϕ^4. The thermodynamic average (28) of the odd terms vanishes. We denote by $K_a \equiv< \phi_a^2 >$, $K_{12} \equiv< \phi_1\phi_2 >$ the Green's correlation functions, and we calculate $< \phi_a^4 >= 3K_a^2$; $< \phi_a\phi_b^3 >= 3K_{12}K_b^2$; $< \phi_a^2\phi_b^2 >= 2K_{12}^2 + K_1K_2$.
Knowing V_0, the explicit expressions for K_a and K_{12} can be deduced from

$$K_a = 2\frac{\partial V_0}{\partial m_a^2}, \qquad K_{12}=\frac{\partial V_0}{\partial \alpha}, \qquad (32)$$

where the definition (28) has been used.

Substituting the thermodynamic averages in (30) leads to:

$$\begin{aligned}G(m_1,m_2,\alpha) &= V_0+\left(\frac{m^2-m_1^2}{2}\right)K_1\\&+\left(\frac{m^2-m_2^2}{2}\right)K_2+\left(\frac{m^2-\mu^2}{2}\right)(\sigma_1^2+\sigma_2^2)\\&+\frac{\lambda}{4!}\left(3K_1^2+6K_1\sigma_1^2+\sigma_1^4+3K_2^2+6K_2\sigma_2^2+\sigma_2^4\right.\\&+2K_1K_2+4K_{12}^2\\&+2K_1\sigma_2^2+2K_2\sigma_1^2\\&\left.+2\sigma_1^2\sigma_2^2+8\sigma_1\sigma_2K_{12}\right)-\alpha K_{12}. \qquad (33)\end{aligned}$$

Notice from (30) that the set G approaches $V(\sigma,\mu)$ by its superior value. We define the GEP $\bar{G}$ as the minimum of G with respect to the three effective parameters determining $\mathcal{L}^{\mathrm{E}}_{\mathrm{free}}$ of (12):

$$\frac{\partial G}{\partial m_1^2}=\frac{\partial G}{\partial m_2^2}=\frac{\partial G}{\partial \alpha}=0. \qquad (34)$$

Equation (34) is in fact a set of three self-consistent equations that determines implicitly the functions $m_a^2(\sigma_a,T)$, $\alpha(\sigma_a,T)$. After some algebra, it can be written as

$$\begin{cases} m_1^2 = m^2+\frac{\lambda}{12}(6K_1+6\sigma_1^2+2K_2+2\sigma_2^2)\\ m_2^2 = m^2+\frac{\lambda}{12}(6K_2+6\sigma_2^2+2K_1+2\sigma_1^2)\\ \alpha \ = \frac{\lambda}{3}(K_{12}+\sigma_1\sigma_2).\end{cases} \qquad (35)$$

In a detailed study [17], we have rigorously shown that the above set implies an invariant $\bar{G}$ by rotations in the plane $[\sigma_1,\sigma_2]$. Thus, one can choose a preferred direction in that plane, for example $\sigma_2 = 0$, $\sigma_1 \equiv \sigma$. From this choice, α, K_{12} are easily evaluated at zero, and the set is reduced to:

$$\begin{cases} m_1^2 = m^2+\frac{\lambda}{12}(6K_1+6\sigma^2+2K_2)\\ m_2^2 = m^2+\frac{\lambda}{12}(6K_2+2K_1+2\sigma^2).\end{cases} \qquad (36)$$

Substituting $\alpha = 0$, $K_{12} = 0$ into (33) and inserting the above implicit functions $m_1^2(\sigma,T)$, $m_2^2(\sigma,T)$, gives:

$$\begin{aligned}\bar{G} &= V_0+\left(\frac{m^2-m_1^2}{2}\right)K_1+\left(\frac{m^2-m_2^2}{2}\right)K_2\\&+\left(\frac{m^2-\mu^2}{2}\right)\sigma^2\\&+\frac{\lambda}{4!}(3K_1^2+6K_1\sigma^2+\sigma^4\\&+3K_2^2+2K_1K_2+2K_2\sigma^2). \qquad (37)\end{aligned}$$

It should be emphasized that the set of self-consistent equations (36) is satisfied by all the extrema of G with respect to m_1^2 and m_2^2, and not just the for minima. We must take care to select the right solution from the possible solutions. Furthermore, the global minimum of G might not be a solution of (36), but might occur at one or another end-point of the ranges $0 < m_1, m_2 < \infty$. In such a case, we choose the end-points as solutions for the effective parameters.

Finally, making use of (32), the correlation functions are:

$$K_a = \int \frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{\partial E_+}{\partial m_a^2} + \frac{\partial E_-}{\partial m_a^2}\right) +2\int \frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{\frac{\partial E_+}{\partial m_a^2}}{e^{\beta E_+}-1} + \frac{\frac{\partial E_-}{\partial m_a^2}}{e^{\beta E_-}-1}\right) = K_{av} + K_{aT}. \tag{38}$$

K_a diverge as k^2, due to the first term K_{av}; the second integrals K_{aT} average the thermal fluctuations in the correlation Green's functions and are finite.

3 Results after renormalization

The ϕ^4 theory (8) has two bare parameters: m^2 and λ. In order to have the GEP in manifestly finite form, free of divergent integrals (cf. K_1, K_2 and V_0), we need to reparametrize the theory in terms of a set of two finite parameters. As $m^2/2$ and $\lambda/12$ are obtained by differentiating the original potential once and twice at the origin, with respect to ϕ^2, an obvious strategy would be to use the set m_R^2 and λ_R, defined through differentiating the GEP with respect to σ^2 once and twice, at $\sigma = 0$, $T = 0$ and $q = 0$:

$$\frac{d\bar{G}}{d\sigma^2}|_{\sigma=0,T=0} = \frac{\mu_R^2}{2}$$
$$\frac{d^2\bar{G}}{d\sigma^4}|_{\sigma=0,T=0} = \frac{\lambda_R}{12}. \tag{39}$$

Let us employ dimensional regularization, operating in $(4-\epsilon$, infinitesimal $\epsilon)$ dimensions. We consider $\lambda = \lambda(\epsilon)$, $m^2 = m^2(\epsilon)$; m_1^2, m_2^2 would be also functions of ϵ through self-consistent equations. Furthermore, we expand these parameters in series of ϵ:

$$\lambda = \sum l_n \epsilon^n, \qquad m^2 = \sum m_{0n}^2 \epsilon^n, \qquad m_i^2 = \Sigma m_{in}^2 \epsilon^n. \tag{40}$$

It turns out to be a convenient procedure to ask that the derivative $d\bar{G}/d\sigma^2$ be finite, in order to have a finite GEP. Making use of the self-consistent equations, we obtain then

$$\frac{d\bar{G}}{d\sigma^2}|_{T=0} = \frac{\partial\bar{G}}{\partial\sigma^2}|_{T=0} + \frac{\partial\bar{G}}{\partial m_a^2}\frac{\partial m_a^2}{\partial\sigma^2}|_{T=0} = \frac{\partial\bar{G}}{\partial\sigma^2}|_{T=0} = \frac{m^2}{2} + \frac{\lambda}{4!}(6K_1 + 2K_2 + 2\sigma^2) = \frac{m_1^2}{2} - \frac{\lambda}{6}\sigma^2. \tag{41}$$

The mass parameter m_1^2 must be finite and positive definite (cf. Stevenson [2]); thus λ must not diverge for $\epsilon \to 0$. As we have shown in [17], a finite λ would lead to a trivial, uninteresting case, so we search for an infinitesimal λ: $\lambda = l_1\epsilon$. Let us expand K_{1v}, K_{2v} at $q = 0$ and V_{0v} in series of ϵ (see Appendix):

$$K_{av} = \frac{A_{-1}m_{a0}^2}{\epsilon} - \frac{A_{-1}}{2}m_{a0}^2 \ln\left(\frac{m_{a0}^2}{\Lambda^2}\right) + A_{-1}m_{a1}^2 + O(\epsilon), \tag{42}$$

$$V_{0v} = \frac{A_{-1}m_{a0}^2 m_{a0}^2}{4\epsilon} - \frac{A_{-1}}{8}m_{a0}^4 \ln\left(\frac{m_{a0}^2}{\Lambda^2}\right) + \frac{A_{-1}}{2}m_{a1}^2 m_{a0}^2 + \frac{A_{-1}}{16}m_{a0}^2 m_{a0}^2 + O(\epsilon) \tag{43}$$

where Λ is the constant of the dimensional regularization, and $A_{-1} = -1/8\pi^2$. Combining the same powers of ϵ, one can write the set of the self-consistent equations in each order of ϵ. First, the lower (zeroth) order is

$$\begin{cases} (1 - \frac{l_1 A_{-1}}{2})m_{10}^2 - (l_1 A_{-1}/6)m_{20}^2 = m_{00}^2 \\ (1 - \frac{l_1 A_{-1}}{/}2)m_{20}^2 - \frac{l_1 A_{-1}}{6}m_{10}^2 \quad = m_{00}^2 \end{cases}. \tag{44}$$

There are only two kinds of interesting nontrivial results for the above set of equations, which we discuss below (for a thorough examination of all possible solutions, one can see [17]):

a) For $l_1 = 3/2A_{-1}$, the determinant of (44) vanishes, and we have $m_1^2 = m_2^2 \equiv M^2$; $m_{00}^2 = 0$. We go on examining the first order in ϵ of the set of self-consistent equations (36), in order to find M^2. After a little algebra, making use of expressions (42, 43), we have

$$\begin{cases} \frac{m_{11}^2}{4} - \frac{m_{21}^2}{4} = m_{01}^2 + \frac{3}{4}\sigma^2 - \frac{3A_{-1}}{8}m_{10}^2 \ln\left(\frac{m_{10}^2}{\Lambda^2}\right) - \frac{A_{-1}}{8}m_{20}^2 \ln\left(\frac{m_{20}^2}{\Lambda^2}\right) \\ \frac{m_{21}^2}{4} - \frac{m_{11}^2}{4} = m_{01}^2 + \frac{1}{4}\sigma^2 - \frac{A_{-1}}{8}m_{10}^2 \ln\left(\frac{m_{10}^2}{\Lambda^2}\right) - \frac{3A_{-1}}{8}m_{20}^2 \ln\left(\frac{m_{20}^2}{\Lambda^2}\right) \end{cases}, \tag{45}$$

from which we deduce, at $\epsilon = 0$, the equation

$$\sigma^2 = -2m_{01}^2 + A_{-1}M^2 \ln\left(\frac{M^2}{\Lambda^2}\right). \tag{46}$$

Substituting the values of λ, m^2 in general GEP (37) for $\epsilon = 0$ leads to:

$$\bar{G} = -\frac{A_{-1}M^4}{4}\left[\ln\left(\frac{M^2}{\Lambda^2}\right) - \frac{1}{2}\right] + M^2\left(\frac{\sigma^2 + 2m_{01}^2}{2}\right). \tag{47}$$

As has already been mentioned in Sect. 2.3, (46) applies only for extrema of $\bar{G}$. Selecting the right solution and excluding the maxima, here we examine particularly the divergent term of $G(m_1, m_2)$ in expression (37):

$$-\frac{A_{-1}}{16\epsilon}[m_{10}^2 - m_{20}^2]^2, \tag{48}$$

which is the dominant term, originated from the power ϵ^{-1} of V_0, K_1. This function of m_{10} and m_{20} represents a minimum at $m_{10} = m_{20}$ only for $\epsilon > 0$ (remember the negativity of A_{-1}). Thus we deduce that $\epsilon > 0$ and $\lambda = (3/2A_{-1})\epsilon < 0$. Note here that for $\epsilon \neq 0$ and large σ, the dominant behavior of $\bar{G}(\sigma)$ would be $\lambda\sigma^4 < 0$, an indefinitely decreasing unbounded GEP. We have found a **precarious** solution. The precarious solution which approaches the four-dimensional theory for $\epsilon \to 0^+$ has been mentioned since the first studies of $\lambda\phi^4$ theory in Gaussian approximation. One can find an extensive discussion, particularly for the real fields, in the works of Stevenson [3], where is has been stressed that the precarious solution is unstable for any UV cutoff and becomes infinitely metastable when the cutoff is removed. In our dimensional regularization, the precariousness is the instability in $(4-\epsilon)$ dimensions and the metastability in 4 dimensions[2].

Our claim of renormalization being to achieve a manifestly finite result, we can apply here the renormalization conditions (39) to eliminate Λ and m_{01} in favor of m_R^2 and λ_R. After some algebra, making use also of (46), we have

$$m_R^2 = M^2|_{\sigma=0}, \tag{49}$$

$$\Lambda^2 = m_R^2 e^{1-(6/(A_{-1}\lambda_R))}, \tag{50}$$

and

$$m_{01}^2 = \frac{A_{-1}}{2} m_R^2 \left[1 - \frac{6}{A_{-1}\lambda_R}\right]. \tag{51}$$

Replacing these values in $\bar{G}$ with (47) and on (46), we are immediately led to:

$$\begin{aligned}\bar{G} = &-\frac{A_{-1}M^4}{4}\left[\ln\left(\frac{M^2}{m_R^2}\right) - \frac{3}{2} + \frac{6}{A_{-1}\lambda_R}\right] \\ &+\frac{M^2}{2}\left[\sigma^2 - A_{-1}m_R^2 + \frac{6m_R^2}{\lambda_R}\right]\end{aligned} \tag{52}$$

and to the following "M^2" equation, typically known as a "gap equation" (see [10]):

$$\begin{aligned}\sigma^2 = &A_{-1}M^2\ln\left(\frac{M^2}{m_R^2}\right) + A_{-1}M^2\left(\frac{6}{A_{-1}\lambda_R} - 1\right) \\ &+A_{-1}m_R^2\left[1 - \frac{6}{A_{-1}\lambda_R}\right].\end{aligned} \tag{53}$$

b) For $l_1 = 3/A_{-1}$, the determinant of (44) vanishes, and we have:

$$m_{10}^2 + m_{20}^2 = -2m_{00}^2 \equiv M^2. \tag{54}$$

[2] See also [10] for a similar conclusion in the case of the real fields.

Thus, the effective parameters for $\epsilon = 0$ are limited in the range $0 \le m_1, m_2 \le M$. We can choose one of them, m_1 for example, as a parameter of variation; the other m_2 would directly be $m_2^2 = M^2 - m_1^2$. Let us first discuss the sign of ϵ, as in the previous case. The divergent ϵ^{-1} term of $G(m_1, m_2)$ in expression (37),

$$\frac{A_{-1}}{8\epsilon}(m_{10}^2 + m_{20}^2)(4m_{00}^2 + m_{10}^2 + m_{20}^2), \tag{55}$$

is also a contribution of V_0, K_1, K_2, λK_1^2, λK_2^2 and $\lambda K_1 K_2$ (where $\lambda = 3/A_{-1}$). It is a function of m_{10}^2 and m_{20}^2, which represents a minimum at $m_{10}^2 + m_{20}^2 = -2m_{00}^2$ only for $\epsilon < 0$ (A_{-1} is negative). Thus we have $\epsilon < 0$ and $\lambda = 3/A_{-1}\epsilon > 0$, a solution which arises in more than four dimensions. This case does not correspond to ultraviolet cutoff, and we will not discuss it here.

4 The precarious solution

Let us now discuss in detail the precarious solution referred to the previous section. At finite temperature, and for $q \neq 0$, the GEP would also contain the terms $V_{0T}(m_1 = m_2 = M)$, $K_{1T} = K_{2T} \equiv K_T(m_1 = m_2 = M)$ and $\mu\sigma^2$ [see (37)]. Because of the equality between effective masses, the spectrum (21) can be simplified to

$$E_{\pm} = \sqrt{k^2 + M^2} \pm \mu. \tag{56}$$

In fact, in conditions of charge conservation, we are rather interested in the Helmholtz free energy, which in our Gaussian approximation would be approximated by $\mathcal{F}$, the Legendre transformation of the GEP, given by the change $\mu \to q$: $\mathcal{F} = \bar{G} + \mu q$, where $q = -\partial\bar{G}/\partial\mu$. Its expression is then:

$$\begin{aligned}\mathcal{F} = &-\frac{A_{-1}M^4}{4}\left[\ln\left(\frac{M^2}{m_R^2}\right) - \frac{3}{2} + \frac{6}{\lambda_R A_{-1}}\right] \\ &+\frac{M^2}{2}\left[\sigma^2 - A_{-1}m_R^2 + \frac{6m_R^2}{\lambda_R}\right] - \frac{\mu\sigma^2}{2} + \mu q + V_{0T},\end{aligned} \tag{57}$$

where the effective mass M is determined by the gap equation at $T \neq 0$:

$$\begin{aligned}\sigma^2 = &A_{-1}M^2\ln\left(\frac{M^2}{m_R^2}\right) + A_{-1}M^2\left(\frac{6}{A_{-1}\lambda_R} - 1\right) \\ &+A_{-1}m_R^2\left[1 - \frac{6}{A_{-1}\lambda_R}\right] - K_T.\end{aligned} \tag{58}$$

The precarious solution has been studied by several authors in the real (noncharged) case. One can find extensive discussions, with or without consideration of thermal effects, in [3, 5, 10]. Here we focus on modifications implied by the background charge, especially those present below the critical temperature.

Making use of the gap equation, we find that the charge of the system is composed of two components:

$$q = -\frac{\partial\bar{G}}{\partial\mu} = -\frac{\partial V_{0T}}{\partial\mu} + \mu\sigma^2. \tag{59}$$

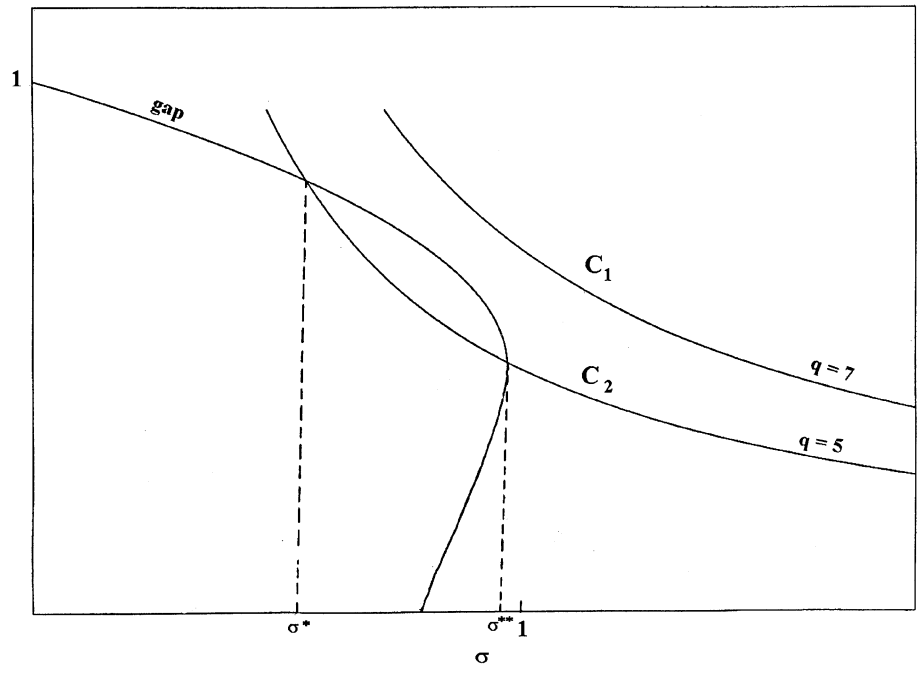

Fig. 1. Functions $M(\sigma)$ and $\mu(\sigma)$ found numerically. $\mu(\sigma)$ is represented by two hyperbola C_1 and C_2 calculated respectively for $q = 7$ and $q = 5$. $M(\sigma)$ is the solution of the gap equation for $\mu_R = 1$, $\lambda_R = -32\pi^2$. The intersection points between the gaps and the hyperbola curves are denoted by σ^* and σ^{**}. We use the $\hbar = c = 1$ units; all values are expressed in MeV

The first component $(-\partial V_{0T}/\partial\mu)$ is the thermal excited charge, while the second one is the charge condensed in the vacuum state.

We employ the expression (57), with $M(\sigma)$ determined by (58) and restricted by relation (59), in order to have the Helmholtz free energy as a function of σ. The chemical potential μ is also determined by the relation (59). Due to complicated integral functions of temperature V_{0T} and K_T, our calculation is a laborious numerical analysis. We represent here the most important results of this study.

In fact, for low temperatures we have $\mu\sigma^2 \gg (-\partial V_{0T}/\partial\mu)$, but the thermal excited charge grows rapidly with the temperature increase. At $T = T_{\text{crit}}$, it would include all the initial background charge. For temperatures above T_{crit}, the thermodynamic potentials and equations would not be markedly different, quantitatively, from the real noncharged case.

So, let us concentrate on solutions below the critical temperature; we begin with $T = 0$.

At $T = 0$, the thermal excited charge vanishes and all the charge is condensed in the vacuum state, $q = \mu\sigma^2$. We have, then, a simple relation between μ and σ values, shown graphically by the two hyperbolas C_1, C_2 in Fig. 1, for two different values of q. Hyperbola C_1 corresponds to a greater charge.

On the other hand, the gap equation (58) determines the solution $M(\sigma)$ represented by the closed curve of Fig. 1. Remember that the (58) comes from $\mathrm{d}\mathcal{F}/\mathrm{d}M = 0$, so the closed curve contains these M which realize extrema of $\mathcal{F}(M)$. As $\mu \leq M$, by definition, the lowest acceptable M is the so-called end-point $M_e = q/\sigma^2$. If the extrema of $\mathcal{F}(M)$ occur at $M \leq M_e$, they are not accepted since they are located below the lowest permitted value of M.

With the condition $\mu \leq M$, the charge curve (C_1 and C_2) separates the plane $[M, \sigma]$ into two parts. The above part is for permitted values of M, and the other for non-permitted ones. Let us go on comparing the hyperbola with the curve $M(\sigma)$. The first one depends on q, the second one on m_R^2 and λ_R. So, there are always two possibilities:

1. The two curves do not intersect (as in the case of C_1). In such a case, for all values of σ, $\mathcal{F}(M)$ is limited on the left by an end-point M_e located over those values of M, which realize the extrema given by the gap equation. Therefore, $\mathcal{F}(M)$ would always be an increasing function of M, with its lowest value situated at the end-point $M_e = q/\sigma^2$, which is very large for $\sigma \to 0$ and vanishing for $\sigma \to \infty$. With $M = M_e/\sigma^2$, the function $\mathcal{F}(\sigma)$ of (57) increases indefinitely for $\sigma \to 0$ and vanishes asymptotically for $\sigma \to \infty$. It is shown by the decreasing curve e_1 of Fig. 2. In the noncharged case, the analogue of e_1 is the "plateau" (see [10]).

2. The two curves intersect (as in the case of C_2). In such a case, for some values of σ one could find solutions of the gap equation (58) that fulfill the charge condition. So, the function $\mathcal{F}(\sigma)$ could possess a minimum, as is given in Fig. 2 by the curve with two branches, e_1 and e_2 (which show, respectively, the behaviour $\mathcal{F}(\sigma)$ for $\sigma \to \infty$, $M \to 0$, and $\sigma \to 0$, $M \to \infty$). However, for other charge values somewhere between about $q = 5$ and $q = 6$, one would find only one intersection between the hyperbola and the gap equation. In such a case, the minimum appears at the end-point of the function $\mathcal{F}(M)$. For a sufficiently smaller value of q, this point, which becomes a small interval where the portion of $\mathcal{F}$ governed by the gap equation and the e_1, e_2 portions join, would be smoother than in Fig. 2.

There is no maximum on the second curve of the Fig. 2, because C_2 in Fig. 1 intersects the gap curve above M values realizing maxima. For $q < 5$, hyperbola C_2 goes down and intersects the gap curve in such a way that for some

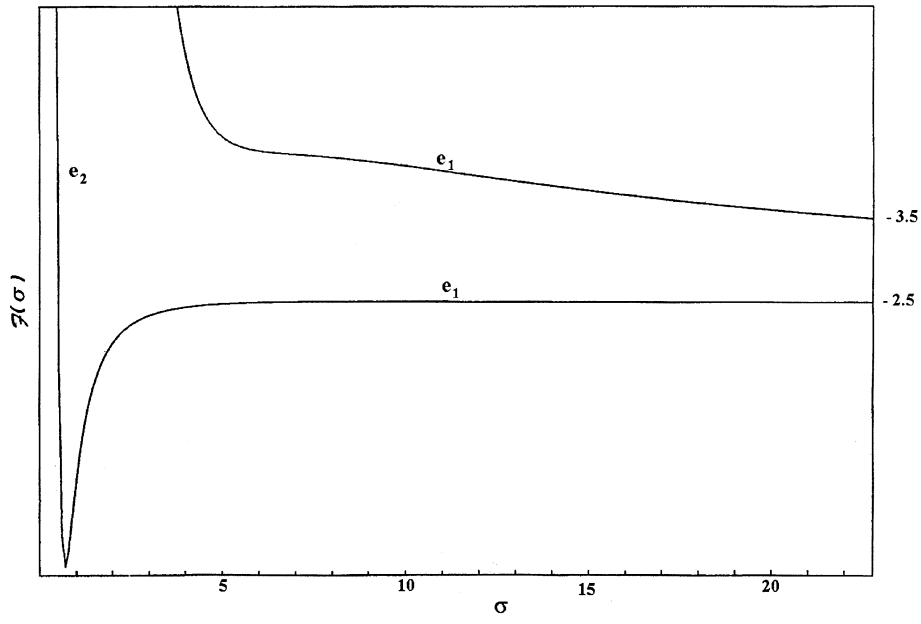

Fig. 2. Helmholtz Gaussian free energy as a function of σ at $T = 0$ for different values of charge, $q = 5$ and $q = 7$; e_1 corresponds to the case of nonintersection between the hyperbola and the gap curve in Fig. 1, so we used $q = 7$. The other curve has two branches, e_1 and e_2, and corresponds to the case of intersection between hyperbola and gap curve in Fig. 1, so we used $q = 5$. The m_R and λ_R values are the same as in Fig. 1. We shifted the two curves vertically to prevent their intersection; their natural vertical positions are not respected

values of σ, one could obtain two solutions of the gap equation; one of them (the smallest M) would be responsible for a maximum on $\mathcal{F}(M)$. This kind of situation is represented in Fig. 3, always for $T = 0$. Nevertheless, the maximum is not an interesting solution.

Now let us discuss the influence of the finite temperature. In such a case, the evaporated (noncondensed) charge will be finite and will influence the gap and charge solutions. The two respective equations (58, 59) now contain terms depending on temperature. For all values of σ, the numerical resolution of these two equations defines the functions $\mu(\sigma)$ and $M(\sigma)$; these are plotted in the Fig. 4 for a fixed value of T. In contrast to the case without temperature, there is a triple solution $\mu(\sigma)$ around σ^{**}. This triple solution stands for the three different values of $\partial V_{0T}/\partial T$: one calculated on the minimum of the gap equation, one on the maximum, and one on the end-point. The origin of the triple solution will be clearer from Fig. 5, where the corresponding $\mathcal{F}(\sigma)$ function is shown. Figure 5 is obtained by replacing expression (57) with the above solutions $M(\sigma)$ and $\mu(\sigma)$. There are four branches corresponding to different situations of the function $\mathcal{F}$ with respect to the gap equation. Branch (1) is found for those values of M which are minimum solutions of the gap equation. On the other hand, the zoomed branch (2) is for the maximum solution of the gap equation. The maximum solution, which occurs at the interval around σ^{**} of Fig. 4, is unstable and uninteresting. The branches denoted by e_1 and e_2 correspond to end-point values of M, for large σ^2 and small σ^2, respectively. The first end-point is known in the real noncharged case, which is mentioned by numerous authors. The second one owes its existence to the charge conservation, and it is absent from the previously mentioned works, in which the minimum of $\mathcal{F}(\sigma)$ is situated at $\sigma = 0$. We stress once more the three different branches in the zoomed part of Fig. 5, minimum, maximum and end-point, that are responsible for the triple μ solution around σ^{**} of Fig. 4.

Notice the minimum $\sigma_{\min} \neq 0$ of the function $\mathcal{F}(\sigma^2)$. The symmetry breaking of the Bose–Einstein-condensation type occurs.

The analogous free case is treated by many authors (see [12, 15, 16]). The symmetry breaking of the Bose-Einstein type has been discovered, due to the charge q. Nevertheless, in our interacting charged case, the critical temperature T_{crit} would depend not only on q and m, as in the free case (see references above), but on λ_R, too. The truth of this statement can be directly seen by the presence of λ_R in (57, 58), which is directly linked to λ by (39) and is an immediate consequence of the interaction studied by our Gaussian approximation. To have a clear idea of the dependence of the critical temperature on λ_R, we develop at large values of temperature and find an equation for T_{crit}:

$$\left(1 + \frac{48\pi^2}{\lambda_R}\right)\left(\frac{9q^2}{8\pi^2 T_{\text{crit}}^4} - \frac{m_R^2}{128\pi^4}\right) = \frac{9q^2}{8\pi^2 T_{\text{crit}}^4} \ln \frac{9q^2}{m_R^2 T_{\text{crit}}^4}. \tag{60}$$

To finish the comparison between the free and the interacting case, we stress that the free curve $\mathcal{F}(\sigma)$ would not possess the left branch e_1, which is a consequence of the end-point considerations.

In Fig. 6, we have plotted the temperature dependence $\sigma_{\min}(T)$, calculated numerically. At $T = 0$, this value is maximum and it vanishes at the critical temperature T_{crit}. It is clearly seen from the charge equation (59) that at $T > T_{\text{crit}}$, the condensed charge would be evaporated, and the minimum of $\mathcal{F}(\sigma)$ would be at $\sigma_{\min} = 0$. The symmetry is restored, and the behavior of $\mathcal{F}(\sigma)$ would be similar to that of the real case.

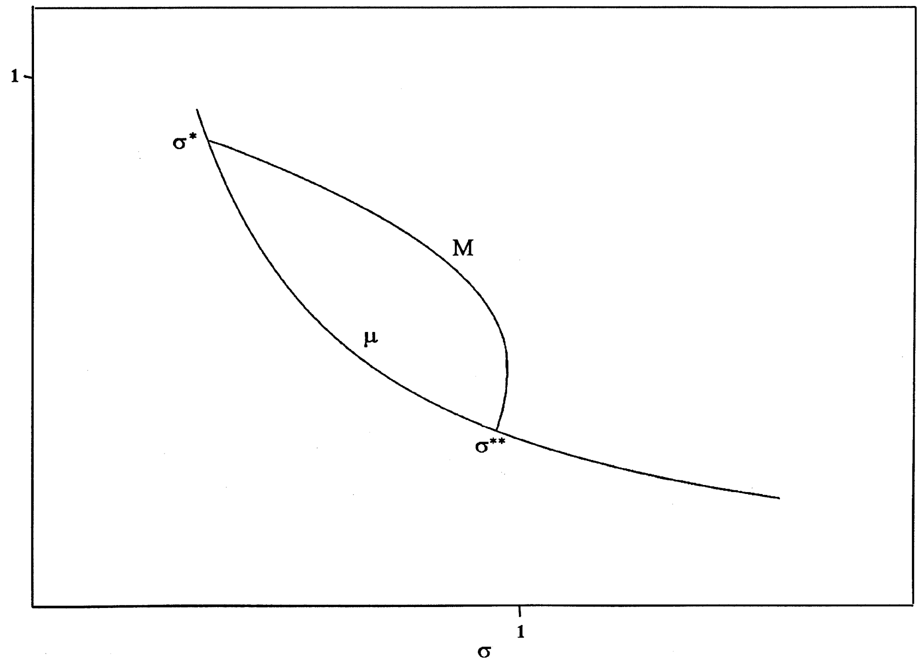

Fig. 3. Functions $M(\sigma)$ and $\mu(\sigma)$ at $T = 0$, $q = 0.3$. The other values of parameters are the same as in Fig. 1

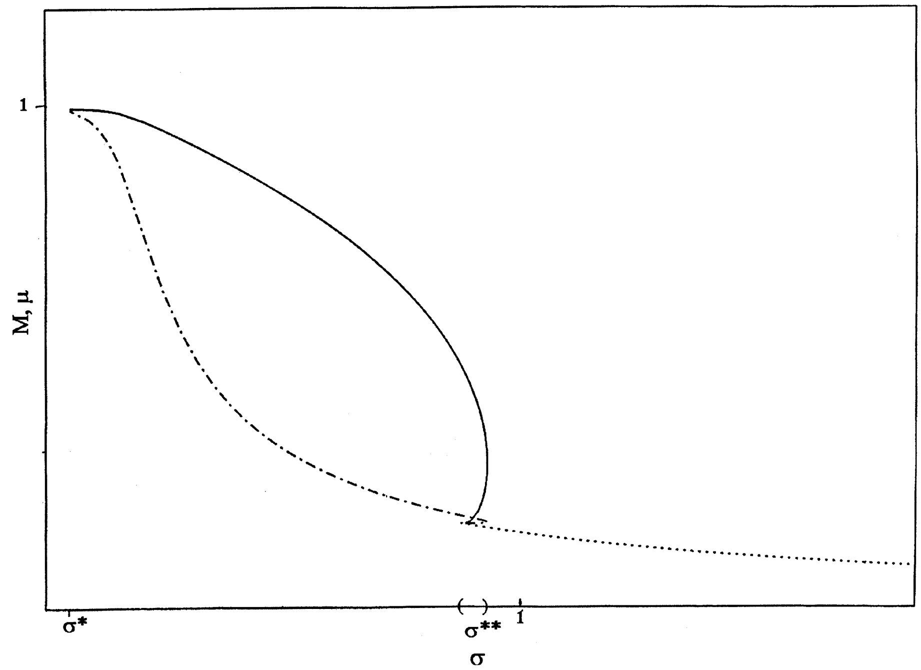

Fig. 4. Functions $M(\sigma)$ and $\mu(\sigma)$ at finite temperature ($T = 0.1$, q=0.7). The solid curve, $M(\sigma)$, is the minimum solution of the gap equation; it stands for $\mu_R = 1$ and $\lambda_R = -32\pi^2$. The dashed curve, $\mu(\sigma)$, is the solution of the charge equation and stands for the same values of parameters. The dotted curve, $M(\sigma) = \mu(\sigma)$, stands for end-point solutions. The curve joining the dashed and the dotted ones is responsible for the maximum solutions of the gap equation

5 Appendix

We need to calculate, in $(4-\epsilon)$ dimensions, the following integrals:

$$K_{av}|_{q=0} = \int \frac{\mathrm{d}^{3-\epsilon}\vec{k}}{2(2\pi)^3} \frac{1}{(\vec{k}^2 + m_a^2)^{1/2}} = A(\Lambda, \epsilon) m_a^{2-\epsilon} \quad (61)$$

and

$$V_{0v} = \int \frac{\mathrm{d}^{3-\epsilon}k}{2(2\pi)^3} \left[(\vec{k}^2 + m_1^2)^{1/2} + (\vec{k}^2 + m_2^2)^{1/2} \right] = \frac{A(\Lambda, \epsilon)}{4-\epsilon} (m_1^{4-\epsilon} + m_2^{4-\epsilon}) \quad (62)$$

where the coefficient A, which contains the divergence, is (see [10]):

$$A(\Lambda, \epsilon) = \frac{\Lambda^{\epsilon/2}}{4\pi^{2+\epsilon/2}} \frac{\Gamma(1+\epsilon/2)}{\epsilon(-1+\epsilon/2)} = A_{-1}\left(\frac{1}{\epsilon} + \ln \Lambda\right) + O(\epsilon). \quad (63)$$

Here Λ is the constant of dimensional regularization, and $A_{-1} = -1/8\pi^2$. At the same time, we calculate

$$m_a^{n-p\epsilon} = (m_a^2)^{n/2-p\epsilon/2} = (m_{a0}^2 + m_{a1}^2\epsilon)^{n/2-p\epsilon/2}$$

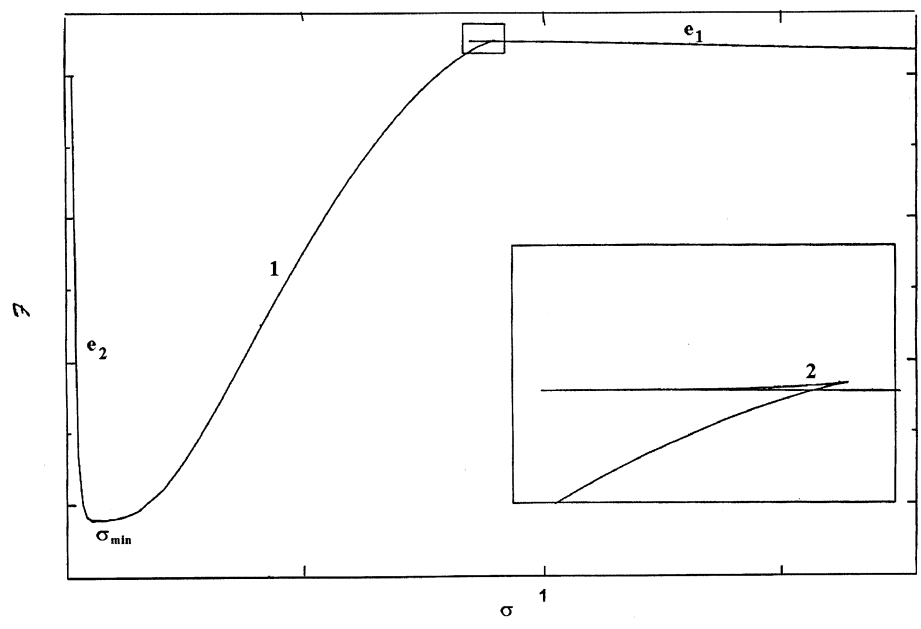

Fig. 5. Gaussian Helmholtz free energy corresponding to solutions of Fig. 4, where the line (1) corresponds to minima given by the gap equation. On the other hand, the zoomed line (2) is the unstable solution, composed by the maxima of the gap equation. The lines e_1 and e_2 correspond to end-points. Note the minimum at $\sigma_{\min} \neq 0$

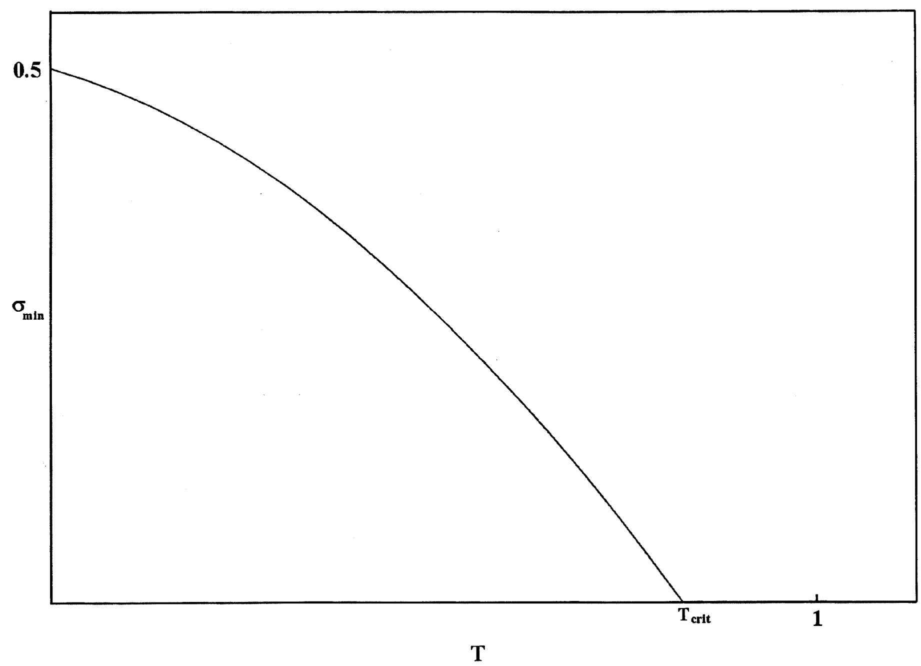

Fig. 6. Function $\sigma_{\min}(T)$, where $\sigma_{\min}$ is the minimum of $\mathcal{F}(\sigma)$ for each value of temperature. This $\sigma_{\min}$ represents the symmetry breaking of the Bose–Einstein-condensation type. The critical temperature indicates the value of temperature where $\sigma_{\min}$ vanishes.
This curve has been realized for $m_R = 1$, $\lambda_R = -32\pi^2$, and $q = 0.7$. The critical temperature occurs at $T = 0.8\,\mathrm{MeV}$

$$= (m_{a0})^{n-p\epsilon}\left[1 + \frac{nm_{a1}^2}{2m_{a0}^2}\epsilon + O(\epsilon^2)\right] \tag{64}$$

$$= m_{a0}^n\left(1 - \frac{p\epsilon}{2}\ln m_{a0}^2\right)\left(1 + \frac{nm_{a1}^2}{2m_{a0}^2}\epsilon\right) + O(\epsilon^2).$$

Replacing the results (63) and (65), respectively, for $A(\Lambda, \epsilon)$ and $m_a^{n-p\epsilon}$, in $K_{av}|_{q=0}$ and V_{0v}, we find the results of expansion in the ϵ power for $K_{av}|_{q=0}$ (42) and V_{0v} (43).

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